A computer-assisted proof that $a(17) = 117$ for OEIS A399997

1 The construction and the search range

Choose 17 integers from $\{0, \dots, 136\}$. How many different positive differences can their pairs give? Write $\Delta(S) = \{y - x : x, y \in S,\ x < y\}$ and $D(S) = |\Delta(S)|$. The largest possible $D(S)$ is $a(17)$ in OEIS A399997. The entry lists the following construction by Ruud H.G. van Tol.

$$S_0 = \{0, 1, 2, 17, 22, 38, 46, 51, 63, 69, 94, 103, 122, 126, 129, 133, 136\}.$$

Counting its pairwise differences gives $D(S_0) = 117$.

It remains to rule out 118 or more. Translate any candidate so its smallest point is 0 and its largest is L. Its positive differences lie in $\{1, \dots, L\}$, so only $118 \le L \le 136$ need checking. A candidate can miss at most $L - 118$ distances and can have at most 18 repeated occurrences among its 136 pairs.

2 Integer Fourier bound

To prune the search, we use an inequality that weights repeated and missing distances. Let r_d be the number of pairs at distance d, $H = \{d : r_d = 0\}$, and $e_d = \max(0, r_d - 1)$. Put $c_d = \cos(d\theta)$. In the squared modulus below, the diagonal terms contribute $17/2$ and each unordered pair at distance d contributes $\cos(d\theta)$.

$$0 \le \tfrac{1}{2}\left|\sum_{x \in S} e^{ix\theta}\right|^2 = \tfrac{17}{2} + \sum_{d=1}^{L} r_d c_d.$$

Using $r_d = 1 - \mathbf{1}_{d \in H} + e_d$ and $\sum_d e_d = 136 - L + |H|$, we obtain

$$\sum_{d=1}^{L} e_d(1 - c_d) \le \tfrac{17}{2} + \sum_{d=1}^{L} c_d + (136 - L) + \sum_{d \in H}(1 - c_d).$$

The search takes $\theta = m_L/10000$, where

$$(m_{118}, \dots, m_{123}) = (380, 376, 373, 370, 367, 364), \qquad m_L = \operatorname{round}\left(\frac{89800}{2L+1}\right) \quad (124 \le L \le 136).$$

Here $|d\theta| < 5$, so the cosine terms decrease in magnitude from index 2 onward. The partial sums through indices 17 and 16 give lower and upper bounds on c_d. We multiply these rational bounds by $M = 10^6$ and round outward to get integers $\ell_d \le M c_d \le u_d$. Set

$$w_d = M - \min(M, u_d), \qquad g_d = M - \ell_d, \qquad B = \tfrac{17M}{2} + \sum_{d=1}^{L} u_d + (136 - L)M.$$

Every candidate must satisfy

$$\sum_{d=1}^{L} e_d w_d \le B + \sum_{d \in H} g_d. \tag{1}$$

3 Endpoint search

At step $t = 1, \ldots, \lfloor L/2 \rfloor$, the search decides whether to include t and $L - t$. The four choices in order are neither, right only, left only, and both. An even midpoint has just two choices. Reflection preserves $D(S)$, so the search keeps the right-only choice at the first asymmetric step and retains all symmetric sets.

After step t, the distance $L - t$ is settled. Any pair $x < y$ with $y - x = L - t$ satisfies $x + (L - y) = t$, so both endpoint offsets are at most t and both positions have been decided.

At a partial set P, let $V = \Delta(P)$, $q = 17 - |P|$, and let h be how many more distances may be missing. Each test below gives a condition that any completion to 118 distinct differences must meet.

- **Cardinality and counts.** There must be room for q more marks. At most $L - 118$ settled distances may be missing, and $\binom{|P|}{2} - |V| \le 18$ since subsequent pairs cannot undo repeats.
- **Fourier budget.** In (1), subtract incurred repeat costs from B and add gains from settled holes. The remaining gains sum to at most the h largest gains among unsettled distances. The sharper test takes them only from unsettled distances outside V. If the balance is still negative, no completion can satisfy (1).
- **Parity.** A completion with u even marks has $u(17 - u)$ odd-difference pairs and $136 - u(17 - u)$ even-difference pairs. For each parity, its distinct count is bounded by both the number of distances left after settled holes and the number of pairs minus repeats already incurred. The sum of the two parity bounds must reach 118 for some attainable u.
- **Future repeats.** Let V_x be the distances from candidate x to P, and put $r(x) = |P| - |V_x \setminus V|$. If X completes P, the pairs from X to P add at least $\sum_{x \in X} r(x)$ repeats, since the union of their new distances has size at most $\sum_{x \in X} |V_x \setminus V|$. Let $s = 18 - \left(\binom{|P|}{2} - |V|\right)$ be the remaining budget. Discard candidates with cost above s, and write the remaining costs in increasing order as $k_1, k_2, \ldots$. A completion requires at least q candidates and $\sum_{i=1}^{q} k_i \le s$. The $q - 1$ cheapest candidates give a lower bound on the cost of any $q - 1$ partners, even if that cheapest group includes x. So a completion containing x requires

$$r(x) \le s - \sum_{i=1}^{q-1} k_i.$$

  The code calls this cutoff `maximum`.
- **Reachable distances.** Let b be the diameter of the eligible positions. A difference above b must occur in V or in some V_x. The number of such differences plus b must reach 118.

Each new pair sets the bit for its difference. At a complete set the number of set bits is $D(S)$.

4 Exhaustive coverage

The search ran as one job per span from 118 to 127. For 128 and 129, each job fixed the first four steps. For spans 130 through 136, jobs fixed the first five steps. At every span the jobs covered all choices. A job that reached its node limit was split into all four next choices. At depth nine the jobs ran without a node limit.

The verifier checked complete coverage of all 7168 roots. The table counts node visits in the completed leaf searches.

Span	Roots	Excluded leaves	Leaf node visits
130	1,024	1,045	20,458,933,920
131	1,024	1,201	53,065,411,204
132	1,024	1,672	125,328,976,456
133	1,024	2,821	293,478,560,412
134	1,024	4,633	626,182,405,342
135	1,024	6,682	1,331,609,556,218
136	1,024	8,389	2,590,441,832,970

Theorem 1. *For OEIS A399997, $a(17) = 117$.*

Proof. The searches found no set with $D(S) \ge 118$ at any span from 118 through 136. By the span reduction, this gives $a(17) \le 117$. The construction S_0 attains 117. □